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ABSTRACT

Behavioral scientists often wish to determine if a sample has been taken from a symmetric population. Similarly, classroom teachers are interested in symmetry if they wish to grade on a "curve." Previously, the sign test, the Wilcoxon test and the t-test have been used to test a hypothesis concerning the symmetry of a distribution of scores about a location parameter. Another test, which is more powerful than either the sign test (S) or the Wilcoxon test (W), and as powerful as the t-test, when the assumptions under the t are met, is the absolute normal scores test (K). The test utilizes the expected order statistics for a sample of absolute values from the standardized normal distribution. The procedure for utilizing the absolute normal scores test is carefully outlined. Two hypothetical examples are presented and analyzed by each of the three tests, S, W, and K. The power of the three nonparametric tests when used with various types of frequency distributions is discussed. (CK)

The Absolute Normal Scores Test for Symmetry

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INTRODUCTION

Behavioral researchers often find it of interest to determine whether or not a sample has been drawn from a symmetric population. This interest in symmetry is also found among those classroom teachers who grade on a curve and assume that examination grades are symmetrically distributed.

In the past, tests such as the sign test, the Wilcoxon test and the t-test have been used to test a hypothesis concerning the symmetry of a distribution of scores about a location parameter designated as Θ. The alternative hypothesis has been that the distribution is asymmetric about Θ.

A test which is equally as powerful as the t-test when the assumptions under the t are met and, in general, more powerful than either the sign test (S) or the Wilcoxon test (W) is the absolute normal scores test (K). Instead of being based upon the sign of the observation (sign test) or the signed rank of the observation (Wilcoxon test), the absolute normal scores test utilizes the expected order statistics for a sample of absolute values from the standardized normal distribution.

PROCEDURES FOR UTILIZING THE ABSOLUTE NORMAL SCORES TEST

Let $X_1, X_2, \ldots, X_n$ represent a random sample of observations drawn from a continuous population whose median is designated as Θ. Form the difference scores, $X_i - \Theta$. For ease in future computations, rank order the absolute value of $X_i - \Theta$ from low to high. Using an indicator variable,

V_i, let $V_i = 1$ if $X_i - \theta > 0$ and $V_i = 0$ if $X_i - \theta < 0$. This indicator variable is useful in setting up the test statistics for the sign test (S), the Wilcoxon test (W) and the absolute normal scores test (K). In each instance the hypothesis under test (H_0) states that the distribution of scores (F) is symmetric about θ. Under H_0 it should be true that the scores will divide themselves evenly above and below the location parameter, θ. This implies that $P(X_i-\theta>0) = P(X_i-\theta<0) = 0.5$. The alternate hypothesis (H_1) being that F is asymmetrical.

The test statistic for the three tests can be written as follows:

(i) $$S = \sum_{i=1}^{n} V_i$$

(ii) $$W = \sum_{i=1}^{n} iV_i$$ where i represents the rank of an individual's absolute difference score within the sample.

(iii) $$K = \sum_{i=1}^{n} E\{|z|_i\}V_i$$ where $E\{|z|_i\}$ is the expected value of the i^{th} absolute normal order statistic in a sample of size n.

A table of the values of $E\{|z|_i\}$ for samples of size 1(1)20, 30, 40, 50, 60, 70, 80, 90, 100 has been published by Govindarajulu and Eisenstat (1965). Critical values of the test statistic, K, at selected significance levels (α) for sample sizes from 1(1)20 have been computed by Klotz (1963) and Thompson, Govindarajulu and Doksum (1967). Critical values for S and W can be found in Owen's (1962) Handbook of Statistical Tables.

For purposes of illustration Example I is analyzed by means of (a) the sign test, (b) the Wilcoxon test and (c) the absolute normal scores test. It is assumed that the rationale for the sign and Wilcoxon tests are accessible to the reader. Detailed descriptions of the two tests can be found in Siegel (1956) and Gibbons (1971). Following these exact procedures, the large sample approximation of the absolute normal scores test to the normal distribution is applied in Example II. When n > 20, normal theory provides for an excellent approximation to the exact test. The test statistic then becomes:

$$Z = \frac{K-E(K)}{\sqrt{Var(K)}}$$

where $$E(K) = \frac{n}{\sqrt{2\pi}}$$

$$Var(K) = \sum_{i=1}^{n} \frac{(E\{|z|_i\})^2}{4}$$

Values of E(K) and Var(K) are presented in Table I for sample sizes 10(1)20, 30, 40, 50, 60, 70, 80, 90, 100.

[Insert Table I here]

The absolute normal scores test can be used to (1) determine whether a sample is representative of a population known to be symmetric around a median Θ or (2) test for the symmetry of a population around Θ using sample data. Example I will illustrate the former situation, Example II illustrates the latter.

HYPOTHETICAL EXAMPLE I

In a beginning course in educational statistics, students were given training in determining the relationship between two variables by substituting into the computing formula for the Pearson Product Moment Correlation Coefficient (r). Following the training, each student was asked to compute r for a problem presented by the instructor. The length of time (in minutes) needed to complete the computations was recorded. These times were ordered from low to high and are presented below:

Subject	Time	Subject	Time
1	18	8	14
2	18	9	13
3	18	10	26
4	17	11	26
5	17	12	26
6	22	13	29
7	23	14	33

From previous research conducted by the instructor, the median computation time was found to be 19 minutes. The instructor now wishes to know whether the distribution of completion times is symmetrical about the median (Θ).

If the median ($\Theta = 19$) is selected as the measure of central tendency about which the time scores are distributed, then subtract the median (Θ) from each score and form deviation scores.

For these data, can it be concluded that the scores are drawn from a population which is symmetric about Θ? Let the probability of a Type I error be 0.05 or less. Information pertinent for analyzing this problem by the various tests of interest is presented in Table II.

[Insert Table II here]

<u>Sign Test (S)</u>

This is one of the oldest known statistical tests described in the printed literature. Dr. John Arbuthnott (1710) proposed its use as a means of testing his hypothesis that the male birth rate is greater than the female birth rate. He attributed this variation to Divine Providence. It is a very simple test since it utilizes only the sign of the difference score, $T_i - \Theta$. The customary procedure is to assign those differences with a positive sign the number 1 and those with a negative sign the number 0. For the example being presented in this paper, the set of 0's and 1's will be labeled V_i. The test statistic for the sign test is

$$S = \sum_{i=1}^{14} V_i = 7.$$

From tables in Owen's (1962) Handbook, the hypothesis that the time scores are symmetrical will be rejected for a two-tailed test at the 0.05 level when S is greater than or equal to 12. Thus the hypothesis is not rejected.

If the probability of a positive or negative sign is assumed to be equally likely, then the exact probability of 12 or more positive differences is given by

$$P = \sum_{i=12}^{14} \binom{14}{i}\left(\tfrac{1}{2}\right)^{14} = \binom{14}{12}\left(\tfrac{1}{2}\right)^{14} + \binom{14}{13}\left(\tfrac{1}{2}\right)^{14} + \binom{14}{14}\left(\tfrac{1}{2}\right)^{14} = .006$$

Doubling this value for a two-tailed test gives an exact probability of 0.012. If we increase the critical region to include an S of 11, the probability of a Type I error will exceed our previously set value of 0.05.

Wilcoxon Test (W)

This test is not only concerned with the sign of the difference score, $T_i - \theta$, but also the magnitude of this difference. With the utilization of this added bit of information, the Wilcoxon Test is, in most instances, more powerful than the sign test. To compute the test statistic, rank order the absolute values of the difference scores, $T_i - \theta$, and then sum those ranks associated with positive differences. Using the notation developed in Table II, the test statistic becomes $W = \sum_{i=1}^{14} iV_i = 73$.

Once again consulting Owen's (1962) Handbook, for a two-tailed test, H_0 will be rejected at the 0.05 level if W is greater than or equal to 84. The symmetry hypothesis is not rejected.

Absolute Normal Scores Test (K)

To apply this test simply replace the ranks of the Wilcoxon Test with the corresponding expected absolute normal order statistics ($E\{|z|_i\}$). Sum the $E\{|z|_i\}$'s associated with positive difference scores. The results for Example I are presented in Table II. The test statistic is

$$K = \sum_{i=1}^{14} E\{|z|_i\}V_i = 8.271082$$

$E\{|z|_i\}$ is what you expect the i^{th} ordered score to be in a sample of size N when drawn from a distribution of random absolute normal deviates. The distribution of random absolute normal deviates is referred to as the chi-distribution. For a given sample size (N), values of $E\{|z|_i\}$ have been tabled by Govindarajulu and Eisenstat (1965).

The critical value of K for α = .05 and N = 14 is obtained from tables derived by Thompson, Govindarajulu and Doksum (1967). The decision rule is to reject H_0 if K > 9.083. Once again the hypothesis calling for a symmetric distribution is not rejected.

HYPOTHETICAL EXAMPLE II

Mr. X has constructed his final exam in advanced algebra and once again has decided to grade on a curve. In the past, he has used the following procedure: 10%-A, 20%-B, 40%-C, 20%-D and 10%-E. Since he plans to use the same percentage breakdown for this exam, he is very interested in knowing whether the distribution of exam scores is symmetric. He is afraid that if the distribution of grades is not symmetrical, certain individuals could be graded unfairly. For example, if the distribution is positively skewed, students at the lower end of the distribution would suffer from his curve. Thus, he decides to test the distribution for symmetry before applying the curve. The test is given and the grades and other pertinent information are presented in Table III.

[Insert Table III here]

The instructor is attempting to determine whether or not the exam scores come from a symmetric population. Since the sample size is 30, the large sample approximation to the normal distribution can be applied to the data.

To apply the absolute normal scores test, carry out the following procedures.

a) Determine the median, $\hat{\Theta}$, of the sample scores. For our data, $\hat{\Theta} = 65$.

b) Compute the deviation scores $G_i - \hat{\Theta}$, and rank order the absolute values of $G_i - \hat{\Theta}$ from low to high.

c) Calculate the variable V_i. $V_i = 1$ if $G_i - \hat{\Theta} > 0$ and $V_i = 0$ if $G_i - \hat{\Theta} < 0$.

d) Replace the ranks with the corresponding expected absolute normal scores, $E\{|z|_i\}$.

e) Determine the value of $K = \sum_{i=1}^{30} E\{|z|_i\}V_i$.

When N = 30, the values of E(K) and Var(K) can be obtained from Table I.

$$K = \sum_{i=1}^{30} E\{|z|_i\}V_i = 17.637$$

$E(K) = 11.9683$ $\qquad$ $Var(K) = 7.2592$

The test statistic is

$$Z = \frac{K-E(K)}{\sqrt{Var(K)}} = \frac{17.637 - 11.968}{2.694} = 2.1$$

If a two-tailed test is used, the hypothesis, H_o, will be rejected at the 0.05 level if Z is greater than or equal to 1.96. Since Z = 2.1, the hypothesis concerning the symmetry of scores in the parent population is rejected With this new information the instructor decided to abandon his curve grading procedure.

CONCLUSION

At this point the researcher is probably trying to determine which of the three nonparametric tests (K, W, S) is the best test to use when attempting to access the symmetry of a distribution of scores. From a statistical standpoint this test goodness is referred to as power. The best test is the one with the greatest power. The power of these three tests with respect to specific population distributions has been evaluated by Thompson, Govindarajulu and Doksum (1967).

For normally distributed data, the absolute normal scores test (K) and the Wilcoxon test (W) are equally good, with both being slightly less powerful than the t-test. K is slightly more powerful than W for small shift alternatives.

When the scores are drawn from a uniform (sharp tailed) distribution, K proves to be superior to W for both large and small shift alternatives. On the other hand, when the distribution has sprawling tails (logistic and double exponential distributions), K and W are equally effective in detecting a lack of symmetry. W is slightly more powerful than K with increasing shift and sample size. Generally the sign test is less powerful than either the absolute normal scores or Wilcoxon tests.

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TABLE I

Expected Values, Variances and Standard Deviations for the Absolute Normal Scores Test Statistic (K)

N	E(K)	Var(K)	$\sqrt{Var(K)}$
10	3.9894	2.2975	1.516
11	4.3884	2.5437	1.595
12	4.7873	2.7904	1.670
13	5.1862	3.0374	1.743
14	5.5852	3.2847	1.812
15	5.9841	3.5322	1.879
16	6.3831	3.7799	1.944
17	6.7820	4.0278	2.007
18	7.1810	4.2758	2.068
19	7.5799	4.5239	2.127
20	7.9788	4.7722	2.185
30	11.9683	7.2592	2.694
40	15.9577	9.7507	3.123
50	19.9471	12.2445	3.499
60	23.9365	14.7396	3.839
70	27.9260	17.2357	4.152
80	31.9154	19.7327	4.442
90	35.9048	22.2310	4.715
100	39.8942	24.7272	4.973